# ANALYTICAL DERIVATION OF EXIT CHARTS FOR SIMPLE BLOCK CODES AND FOR LDPC CODES USING INFORMATION COMBINING

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## ABSTRACT

The extrinsic information transfer (EXIT) chart describes the input-output behavior of a decoder by means of the mapping from a-priori information and channel information to extrinsic information. In this paper, we consider single parity check and repetition codes over binary-input symmetric memoryless channels. Using the concept of information combining, we derive bounds on the extrinsic information for these codes, which depend only on the a-priori information and on the channel information, but not on the channel models. The bounds are applied to the EXIT charts of these codes and to the EXIT charts of low-density parity-check codes.

## 1. INTRODUCTION

Extrinsic information transfer (EXIT) charts have shown to be a powerful tool for analysis and design of iteratively decodable channel codes [1, 2]. These charts show for each constituent decoder the mapping from a-priori information to extrinsic information, called EXIT function. The mutual information of the communication channel, denoted as channel information, is used to parameterize the curves.

Since a decoder combines the a-priori information and the channel information to the extrinsic information, decoding may be interpreted as information processing. The combining of mutual information is called *information combining* [3].

For given a-priori information and channel information, the combined information can be computed exactly if models for the a-priori channel and for the communication channel are assumed. Often, the binary-input additive white Gaussian noise (AWGN) channel is applied as channel model (e.g. [1, 2]). On the other hand, bounds on the combined information can be given, if the a-priori channel and the communication channel are only required to be symmetric and memoryless.

The concept of bounding combined information was introduced in [4] for two binary-input symmetric memoryless channels having the same input. In this paper, we generalize this concept to give bounds on the extrinsic information for single parity check codes and for repetition codes. As applications, we consider analytically the EXIT charts of these codes and the EXIT charts of low-density parity-check codes. Using [4, 5] as a starting point, similar concepts were independently developed in [6], recently.

This paper is organized as follows: In Section 2, the decoding model is introduced. In Section 3, bounds on the extrinsic information are derived for single parity check codes and for repetition codes. These bounds are applied in Section 4 to EXIT charts for these codes and to EXIT charts for LDPC codes. Section 5 summarizes our results.

## 2. DECODING MODEL

Throughout this paper, random variables are denoted by uppercase letters, and realizations are denoted by lowercase letters. For a vector $\boldsymbol{a} = [a_1, \ldots, a_J]$, we adopt the short-hand notation $\boldsymbol{a}_{\setminus i} := [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_J]$.

Consider a single parity check code or a repetition code of length $N$ with equiprobable code words $\boldsymbol{x} = [x_0, x_1, \ldots, x_{N-1}] \in \mathbb{B}^N$, $\mathbb{B} := \{-1, +1\}$. The noisy observation of code bit $X_i$ is denoted by $Y_i$, $Y_i \in \mathbb{R}$, and the channel between this code bit and its observation is denoted by $X_i \rightarrow Y_i$, $i = 0, 1, \ldots, N-1$. These channels are assumed to be independent, and they are assumed to be binary-input symmetric memoryless channels. In Fig. 1, the code constraints and the transmission channels $X_i \rightarrow Y_i$ are illustrated for $N = 4$. The mutual information of the channel $X_i \rightarrow Y_i$ is called *intrinsic information* about code bit $X_i$: $I_{\mathsf{int},i} := I(X_i; Y_i)$.

In an iterative decoder, a constituent decoder may see two types of channels. On the one hand, there is the communication channel; its mutual information is called *channel information* $I_{\mathsf{ch}}$. On the other hand, there is the virtual channel between a code bit and the soft estimate, provided by another constituent decoder and used as a-priori value; the mutual information of this "a-priori channel" is called *a-priori information* $I_{\mathsf{apri}}$. Thus, we have $I_{\mathsf{int},i} = I_{\mathsf{ch}}$ if the channel $X_i \rightarrow Y_i$ is the communication channel, and we have $I_{\mathsf{int},i} = I_{\mathsf{apri}}$ if the channel $X_i \rightarrow Y_i$ is the a-priori channel. Let $\mathbb{I}_{\mathsf{apri}}$ denote the index set of the code bits, about which we have a-priori information.

For each code bit $X_i$, $i \in \mathbb{I}_{\mathsf{apri}}$, the decoder computes the extrinsic log-likelihood ratio (LLR) [7, 8]:

$$w_i := \ln \frac{\Pr(X_i = +1 | \boldsymbol{Y}_{\setminus i} = \boldsymbol{y}_{\setminus i})}{\Pr(X_i = -1 | \boldsymbol{Y}_{\setminus i} = \boldsymbol{y}_{\setminus i})},$$

where ln(.) denotes the natural logarithm. Using LLRs, these values can easily be computed using the boxplus operator for

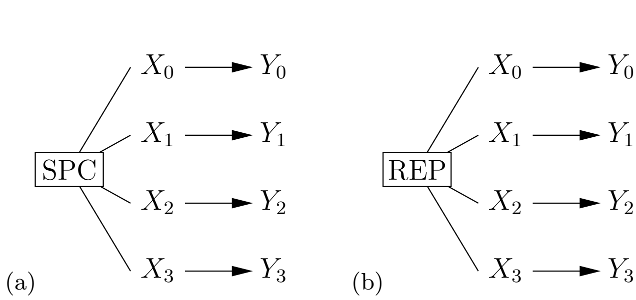


Figure 1: (a) Single parity check (SPC) code and (b) repetition (REP) code of length $N = 4$.

the single parity check code, and using addition for the repetition code [8]. We define the extrinsic information about code bit $X_i$ as

$$I_{\mathsf{ext},i} := I(X_i; W_i).$$

It can be shown that $I(X_i; W_i) = I(X_i; \boldsymbol{Y}_{\setminus i})$, because an extrinsic LLR is basically a special a-posteriori LLR [9]. The average extrinsic information about code bits in $\mathbb{I}_{\mathsf{apri}}$ is called *extrinsic information* [1,2]:

$$I_{\mathsf{ext}} := \frac{1}{|\mathbb{I}_{\mathsf{apri}}|} \sum_{i \in \mathbb{I}_{\mathsf{apri}}} I_{\mathsf{ext},i}.$$

As we consider only single parity check codes and repetition codes, we have obviously $I_{\mathsf{ext}} = I_{\mathsf{ext},i}$ for all $i \in \mathbb{I}_{\mathsf{apri}}$. The *EXIT function* plots the extrinsic information versus the a-priori information. If channel information is available, i.e., if $|\mathbb{I}_{\mathsf{apri}}| \neq N$, it is used to parameterize the EXIT functions.

As mentioned above, we restrict ourselves to *binary-input symmetric memoryless channels* (BISMCs). Examples for BISMCs are the binary symmetric channel (BSC), the binary erasure channel (BEC), and the binary-input AWGN channel. In the following, we shortly summarize definitions and properties of BISMCs given in [4].

Let $X \to Y$ denote a BISMC with input $X \in \mathbb{X} := \mathbb{B}$ and output $Y \in \mathbb{Y} \subseteq \mathbb{R}$. The transition probabilities of the channel are given by $p_{Y|X}(y|x)$, denoting the probability density function for continuous output alphabets and denoting the probability mass function for discrete output alphabets. Since the channel is symmetric, we can assume $p_{Y|X}(y|x) = p_{Y|X}(-y|-x)$ for all $x \in \mathbb{X}$ and for all $y \in \mathbb{Y}$ without significant loss of generality.

A BISMC can be separated into subchannels which are BSCs. For this separation, we define the subchannel indicator $J := |Y|$, where $J \in \mathbb{J} := \{y \in \mathbb{Y} : y \geq 0\}$. Thus, for $j > 0$, the channel $X \to Y|J = j$ is a BSC with crossover probability $\epsilon(j) := p_{Y|X,J}(-j|+1, j)$. For $j = 0$ this channel is a BEC with erasure probability 1; but without loss of generality, we can interpret this channel as a BSC with crossover probability $\epsilon(0) := 1/2$.

As all subchannels are BSCs, the mutual information of the subchannel defined by $J = j$ can be written as

$$I(j) := I(X; Y|J = j) = 1 - h(\epsilon(j)),$$

where $h(\xi) := -\xi \,\mathrm{ld}\, \xi - (1-\xi)\,\mathrm{ld}(1-\xi)$, $\xi \in [0,1]$, denotes the binary entropy function. Let further $h^{-1}(\eta)$, $\eta \in [0,1]$, denote the inverse of $h(\xi)$ for $\xi \in [0, 1/2]$. The mutual information $I$ of the BISMC can be obtained by taking the expectation of the mutual information of the subchannels:

$$\begin{aligned} I &:= I(X;Y) = I(X;Y|J) \\ &= \mathop{\mathrm{E}}_{j\in\mathbb{J}}\{I(X;Y|J=j)\} = \mathop{\mathrm{E}}_{j\in\mathbb{J}}\{I(j)\}. \end{aligned}$$

This concept for separation of BISMCs into BSCs can be generalized to channels with vector-valued outputs. Let $X \to \boldsymbol{Y}$ denote a BISMC with input $X \in \mathbb{X} := \mathbb{B}$ and output $\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_n]$, with $Y_i \in \mathbb{Y}_i \subseteq \mathbb{R}$, $i = 1, 2, \ldots, n$. Using $\mathbb{J}_1 := \{y \in \mathbb{Y}_1 : y \geq 0\}$ and $\mathbb{J} := \mathbb{J}_1 \times \mathbb{Y}_2 \times \mathbb{Y}_3 \times \cdots \times \mathbb{Y}_n$, we define the vector-valued subchannel indicator $\boldsymbol{J} \in \mathbb{J}$ as $\boldsymbol{J} := \boldsymbol{j}$ for $\boldsymbol{y} \in \{\boldsymbol{j}, -\boldsymbol{j}\}$. Then, the crossover probability and the mutual information of each subchannel $X \to \boldsymbol{Y}|\boldsymbol{J} = \boldsymbol{j}$ can be defined in a similar way as above.

The separation of a BISMC into subchannels which are BSCs is applied in the following section.

## 3. BOUNDS ON EXTRINSIC INFORMATION

First, we introduce two functions which allow us to write the bounds in a compact form. For each function we give an interpretation.

**Definition 1**
*Let $\xi_1, \xi_2, \ldots, \xi_n \in [0,1]$, $n \geq 1$. We define the* binary information function for serial concatenation *for $n = 1$ as $f_1^{\mathsf{ser}}(\xi_1) := \xi_1$, for $n = 2$ as*

$$f_2^{\mathsf{ser}}(\xi_1, \xi_2) := 1 - h\big((1-\epsilon_1)\epsilon_2 + \epsilon_1(1-\epsilon_2)\big),$$

*and for $n > 2$ as*

$$f_n^{\mathsf{ser}}(\xi_1, \xi_2, \ldots, \xi_n) := f_2^{\mathsf{ser}}\big(\xi_1, f_{n-1}^{\mathsf{ser}}(\xi_2, \xi_3, \ldots, \xi_n)\big),$$

*where $\epsilon_i := h^{-1}(1 - \xi_i)$ for $i = 1, 2, \ldots, n$.*

The meaning of this function is as follows: Consider $n$ independent BSCs, each having mutual information $I_i$, $i = 1, 2, \ldots, n$. Let these BSCs be serially concatenated such that the output of one channel is equal to the input of the following; let further the input of the first channel be uniformly distributed. Then, the mutual information between the input of the first channel and the output of the last channel is given by $f_n^{\mathsf{ser}}(I_1, I_2, \ldots, I_n)$.

**Definition 2**
*Let $\xi_1, \xi_2, \ldots, \xi_n \in [0,1]$, $n \geq 1$, and let $\boldsymbol{r} = [r_1, r_2, \ldots, r_n]$, $r_i \in \mathbb{B}$. We define the* binary information function for parallel concatenation *as*

$$f_n^{\mathsf{par}}(\xi_1, \xi_2, \ldots, \xi_n) := -\sum_{\boldsymbol{r}\in\mathbb{B}^n} \psi(\boldsymbol{r})\,\mathrm{ld}\,\psi(\boldsymbol{r}) - \sum_{i=1}^{n}(1 - \xi_i)$$

*with $\psi(\boldsymbol{r}) := \frac{1}{2}\Big(\prod_{i=1}^n \varphi_i(r_i) + \prod_{i=1}^n \big(1 - \varphi_i(r_i)\big)\Big)$, where $\varphi_i(r_i) := \epsilon_i$ for $r_i = +1$ and $\varphi_i(r_i) := 1 - \epsilon_i$ for $r_i = -1$, and $\epsilon_i := h^{-1}(1 - \xi_i)$ for $i = 1, 2, \ldots, n$.*

The meaning of this function is as follows: Consider $n$ independent BSCs, each having mutual information $I_i$, $i = 1, 2, \ldots, n$. Let the inputs of these channels be the same (this is called parallel concatenation of these channels), and let the inputs be uniformly distributed. Then, the mutual information between the input and the vector comprising the outputs of all channels is given by $f_n^{\mathsf{par}}(I_1, I_2, \ldots, I_n)$.

For the following two theorems, we use the notation introduced in Section 2. The theorems give bounds on the extrinsic information about code bit $X_0$ using only the intrinsic information about the other code bits. The proofs of the theorems are based on the separation of BISMCs into BSCs. We consider first single parity check (SPC) codes and then repetition codes.

**Theorem 1 (Extrinsic Information for SPC Codes)**
*For a single parity check code of length $N$, the extrinsic information about code bit $X_0$ is bounded as:*

$$\begin{aligned} I_{\mathsf{ext},0} &\geq I_{\mathsf{int},1} I_{\mathsf{int},2} \cdots I_{\mathsf{int},N-1}, \\ I_{\mathsf{ext},0} &\leq f_{N-1}^{\mathsf{ser}}(I_{\mathsf{int},1}, I_{\mathsf{int},2}, \ldots, I_{\mathsf{int},N-1}). \end{aligned}$$

*The lower bound is achieved if all channels are BECs, and the upper bound is achieved if all channels are BSCs.*

The proof of the theorem can be found in [10].

**Theorem 2 (Extrinsic Information for Rep. Codes)**
*For a repetition code of length $N$, the extrinsic information about code bit $X_0$ is bounded as:*

$$\begin{aligned} I_{\mathsf{ext},0} &\geq f_{N-1}^{\mathsf{par}}(I_{\mathsf{int},1}, I_{\mathsf{int},2}, \ldots, I_{\mathsf{int},N-1}), \\ I_{\mathsf{ext},0} &\leq 1 - (1 - I_{\mathsf{int},1})(1 - I_{\mathsf{int},2}) \cdots (1 - I_{\mathsf{int},N-1}). \end{aligned}$$

*The lower bound is achieved if all channels are BSCs, and the upper bound is achieved if all channels are BECs.*

Note that BSCs lead to the lower bound for repetition codes but to the upper bound for single parity check codes; for BECs, the reverse holds.

Due to the limited space, we cannot give the complete proof. Instead, we would like to point out the basic ideas: First, we interpret the channel between code bit $X_0$ and the channel outputs $Y_1, Y_2, \ldots, Y_{N-1}$ as the parallel concatenation of the channel $X_1 \to Y_1$ and the channel $Z \to \boldsymbol{Y}_{[2,N-1]}$, where $X_1 = X_0$ due to the repetition code and $Z := X_0$ by definition. Then, the extrinsic information about code bit $X_0$ is a combination of the intrinsic information about code bit $X_1$, $I_{\mathsf{int},1}$, and the intrinsic information about bit $Z$, $I_{\mathsf{int},Z} := I(Z; \boldsymbol{Y}_{[2,N-1]})$. The channel $X_1 \to Y_1$ is a BISMC by definition, and the channel $Z \to \boldsymbol{Y}_{[2,N-1]}$ can easily be seen to be also a BISMC. Thus, we can apply methods similar to that in [4], but we have to take into account that $Z \to \boldsymbol{Y}_{[2,N-1]}$ is a BISMC with vector-valued output.

## 4. BOUNDS ON EXIT FUNCTIONS

The bounds on the extrinsic information, given in the previous section, depend only on the intrinsic information about the code bits. As discussed in Section 2, for a constituent decoder of an iterative decoder, the intrinsic information about a code bit is either equal to the channel information $I_{\mathsf{ch}}$, or it is equal to the a-priori information $I_{\mathsf{apri}}$. Thus, we can directly apply the bounds to the EXIT charts.

As single parity check codes and repetition codes are (short) block codes, they are usually used as constituent codes in concatenated coding schemes in the following way: For encoding, the whole bit sequence is divided into blocks, and the blocks are encoded separately. For decoding, each block of coded bits corresponding to one code is decoded separately. In the following, we assume that only one type of code (e.g., a $(4,3,2)$ single parity check code) is used at the same time. Then, the decoder for the whole set of codes can be characterized by the decoder for only one code.

### 4.1 Single Parity Check Codes

Consider first single party check codes used as outer codes in a serially concatenated coding scheme. In this case, the decoder for one code operates on a-priori information about all code bits, i.e., $|\mathbb{I}_{\mathsf{apri}}| = N$ and $I_{\mathsf{int},i} = I_{\mathsf{apri}}$ for $i = 0, 1, \ldots, N-1$. Observing that $I_{\mathsf{ext},i} = I_{\mathsf{ext}}$ for $i \in \mathbb{I}_{\mathsf{apri}}$, we have from Theorem 1:

$$\begin{aligned} I_{\mathsf{ext}} &\geq (I_{\mathsf{apri}})^{N-1}, \\ I_{\mathsf{ext}} &\leq f^{\mathsf{ser}}_{N-1}(I_{\mathsf{apri}}, I_{\mathsf{apri}}, \ldots, I_{\mathsf{apri}}). \end{aligned} \tag{1}$$

Consider now single parity check codes used as inner codes in a serially concatenated coding scheme, where only the parity bits $X_{N-1}$ are transmitted over the communication channel. Then, the decoder for one code operates on channel information about code bit $X_{N-1}$, $I_{\mathsf{int},N-1} = I_{\mathsf{ch}}$, and a-priori information about the other code bits, $I_{\mathsf{int},i} = I_{\mathsf{apri}}$ for $i = 0, 1, \ldots, N-2$. Observing that $I_{\mathsf{ext},i} = I_{\mathsf{ext}}$ for $i \in \mathbb{I}_{\mathsf{apri}}$, and using Theorem 1, we get:

$$\begin{aligned} I_{\mathsf{ext}} &\geq I_{\mathsf{ch}} \cdot (I_{\mathsf{apri}})^{N-2}, \\ I_{\mathsf{ext}} &\leq f^{\mathsf{ser}}_{N-1}(I_{\mathsf{ch}}, I_{\mathsf{apri}}, \ldots, I_{\mathsf{apri}}). \end{aligned} \tag{2}$$

The bounds given in (2) are depicted in Fig. 2. Obviously, the extrinsic information cannot become larger than the channel information, even if the a-priori information is equal to 1.

### 4.2 Repetition Codes

Bounds on the extrinsic information for repetition codes can be given by means of Theorem 1. When used as outer codes in a serially concatenated coding scheme, the decoder for one code operates on a-priori information about all code bits.

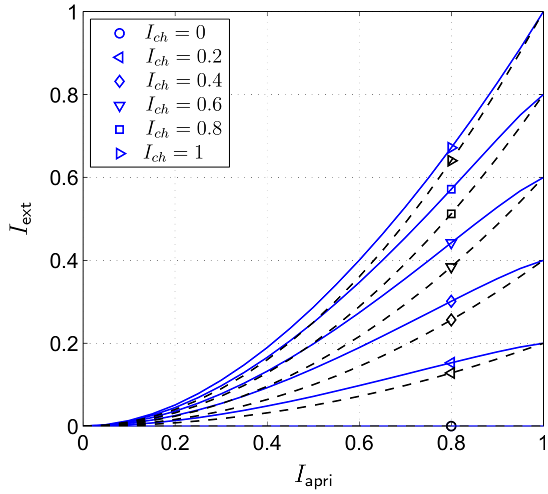


Figure 2: Bounds on the EXIT functions for inner single parity check codes of length $N = 4$ for several values of channel information $I_{\mathsf{ch}}$. (Upper bounds: solid line; lower bounds: dashed line. Mutual information is given in bit/use.)

This is for example the case in nonsystematic repeat accumulate codes [11,12]. Thus, we obtain the bounds:

$$\begin{aligned} I_{\mathsf{ext}} &\geq f^{\mathsf{par}}_{N-1}(I_{\mathsf{apri}}, I_{\mathsf{apri}}, \ldots, I_{\mathsf{apri}}), \\ I_{\mathsf{ext}} &\leq 1 - (1 - I_{\mathsf{apri}})^{N-1}. \end{aligned} \tag{3}$$

The corresponding EXIT charts are depicted in Fig. 3.

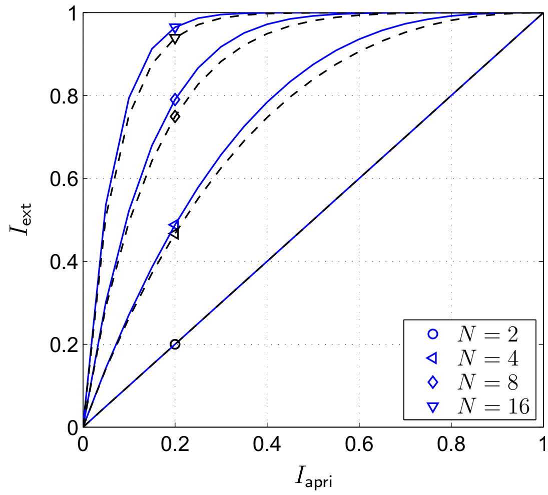


Figure 3: Bounds on the EXIT functions for outer repetition codes of several code lengths $N$. (Upper bounds: solid line; lower bounds: dashed line. Mutual information is given in bit/use.)

In systematic repeat accumulate codes, some code bits of the outer repetition codes are directly transmitted over the communication channel [11,12]. This motivates the following model: Let us assume that the decoder for one code operates on channel information about code bit $X_{N-1}$ and on a-priori information about the other code bits. (This is the same situation as in the case of inner single parity check codes, as discussed above.) Accordingly, the bounds on the extrinsic information are given as:

$$\begin{aligned} I_{\mathsf{ext}} &\geq f^{\mathsf{par}}_{N-1}(I_{\mathsf{ch}}, I_{\mathsf{apri}}, \ldots, I_{\mathsf{apri}}), \\ I_{\mathsf{ext}} &\leq 1 - (1 - I_{\mathsf{ch}})(1 - I_{\mathsf{apri}})^{N-2}. \end{aligned} \tag{4}$$

### 4.3 Low-Density Parity-Check Codes

Low-density parity-check (LDPC) codes (see e.g. [13,14]) are an alternative to parallel and serially concatenated codes (see e.g. [1, 2] and the references therein). Regular LDPC codes are defined by the variable node degree $d_v$ and the check node degree $d_c$, giving the number of ones per column and per row in the parity check matrix, respectively. LDPC codes are decoded using the message passing algorithm on their factor graph, comprising variable nodes and check nodes (see e.g. [15]). The EXIT chart method may be used to determine whether decoding of an LDPC code (of infinite code length) converges for a certain communication channel.

The decoding operation in a variable node is equivalent to computing extrinsic LLRs for a repetition code of length $N = d_v + 1$, where the decoder has channel information about one code bit and a-priori information about the other code bits. Accordingly, we can give bounds on the EXIT functions using (4). Similarly, the decoding operation in a check node is equivalent to computing extrinsic LLRs for a single parity check code of length $N = d_c$, where the decoder has a-priori information about all code bits. Thus, we can give bounds on the EXIT functions using (1).

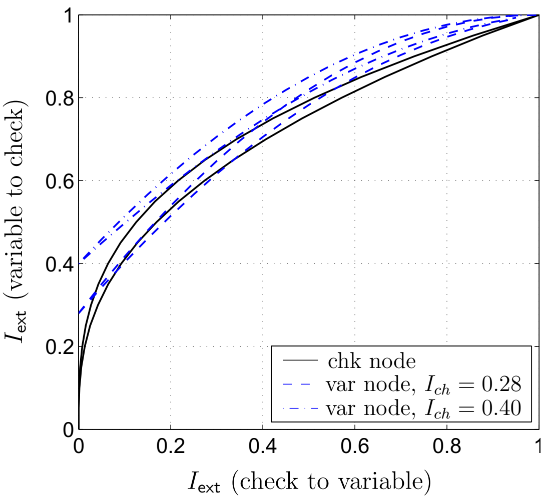


Figure 4: Bounds on the EXIT functions for an LDPC code with $d_v = 3$ and $d_c = 4$. (Mutual information is given in bit/use.)

For illustration, we consider the simple example of an LDPC code with variable node degree $d_v = 3$ and check node degree $d_c = 4$, having design rate $R = 1/4$. The EXIT chart of this code is depicted in Fig. 4 for two values of channel information. (The curves for the check node are flipped.) For $I_{\mathsf{ch}} = 0.28$, the lower bound for the check node EXIT function and the upper bound for the variable node EXIT function touch each other. Thus, the decoder can never converge for $I_{\mathsf{ch}} < 0.28$. Accordingly, this code is not capacity achieving for any communication channel which is a BISMC, because $R < 0.28$. For $I_{\mathsf{ch}} = 0.40$, the upper bound for the check node EXIT function and the lower bound for the variable node EXIT function touch each other. Accordingly, the decoder will converge for sure if $I_{\mathsf{ch}} > 0.40$.

Thus, Theorem 1 and Theorem 2 allow us to give a necessary condition and a sufficient condition for convergence. These conditions are valid for all a-priori channels that are BISMCs and for all communication channels that are BISMCs. On the other hand, the conventional EXIT chart method relies on a specific model for the a-priori channel (often the binary-input AWGN channel is applied), which is usually only an approximation of the actual a-priori channel. Hence, convergence conditions derived from the conventional EXIT chart method cannot guaranteed to be strict.

## 5. SUMMARY

In this paper, we presented bounds on the extrinsic information for single parity check codes and for repetition codes, which are valid for all binary-input symmetric memoryless channels. These were applied to EXIT charts for these codes and for LDPC codes. As opposed to the conventional EXIT chart method, no specific channel models are necessary.

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